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A Static Three-Center Solution to Einstein's Gravitational Field Equations

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Abstract

In this paper the physical situation of three collinear, axisymmetric masses is studied within the framework of Einstein's general theory of relativity. A solution is found, and the feasibility of the existence of coupled negative-positive masses is demonstrated in direct correspondence to such a feasibility in Newtonian theory.

1. *Introduction*

In 1936 Silberstein claimed that he had found a solution to Einstein's gravitational field equations that represented two isolated, axisymmetric masses and that, contrary to "man's most ancient, primitive experience", they did not gravitate towards each other but remained at rest. In the same journal, Einstein and Rosen (1936) contradicted Silberstein's claim and showed that, indeed, the solution did represent two isolated mass centers but they were separated by a strut sufficient to hold the two masses apart and thus produce a static situation. In fact Silberstein had reproduced Curzon's (1924) solution to the same physical problem.

It occurred to the author that a similar static situation could be achieved if the strut were replaced by a body that had the power to repel the two mass centers in such a way as to equally balance their mutual gravitational attraction. Accordingly, following Bondi's (1957) argument, the intermediary body was chosen to be one of negative mass; clearly, in the Newtonian picture three such suitably chosen masses would lie in a straight line without any relative motion.

In Sec. 2 the problem of looking for such a three-center solution is defined and solved, the results being in agreement with Newtonian phenomenological feasibility. In Sec. 3 the author looks to a special case, and in Sec. 4 a discussion of the results is given.

2. *The Problem*

2.1 *Definition*

Given the axisymmetric line element in cylindrical polar coordinates (ρ, z, ϕ, t)

$$ds^2 = e^{2\nu}c^2\, dt^2 - e^{-2\nu}[e^{2\lambda}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] \tag{2.1}$$

it is easily shown that substitution into Einstein's vacuum field equations results in four partial differential equations in the two parameters $\nu(\rho, z)$ and $\lambda(\rho, z)$, namely,

$$\nabla^2 \nu = 0 \tag{2.2}$$

$$\frac{\partial \lambda}{\partial \rho} = \rho\left[\left(\frac{\partial \nu}{\partial \rho}\right)^2 - \left(\frac{\partial \nu}{\partial z}\right)^2\right] \tag{2.3}$$

$$\frac{\partial \lambda}{\partial z} = 2\rho \frac{\partial \nu}{\partial \rho}\frac{\partial \nu}{\partial z} \tag{2.4}$$

$$\frac{\partial^2 \lambda}{\partial \rho^2} + \frac{\partial^2 \lambda}{\partial z^2} + \left(\frac{\partial \nu}{\partial \rho}\right)^2 + \left(\frac{\partial \nu}{\partial z}\right)^2 = 0 \tag{2.5}$$

The problem thus reduces to one where, given the well-known solutions to Eq. (2.2), we substitute into Eqs. (2.3) and (2.4) and integrate. As a check, any solution so obtained must also satisfy Eq. (2.5).

The solution to the metric (2.1) that represents a single isolated axisymmetric body is given as[1]

$$\nu = -Gm/c^2 r, \qquad r^2 = \rho^2 + z^2 \tag{2.6}$$

and

$$\lambda = -G^2 m^2 \rho^2 / 2c^4 r^4 \tag{2.7}$$

m being the mass of the body located at the origin, G the gravitational constant, and c the speed of light in vacuo. In this paper we consider a solution of Eq. (2.2) in the form

$$\nu = -\sum_{i=1}^{3} L_i r_i^{-1} \tag{2.8}$$

where

$$L_i = Gm_i/c^2$$
$$r_i^2 = \rho^2 + (z + \alpha_i)^2 \tag{2.9}$$

2.2 *Integration of Eqs. (2.3) and (2.4)*

Given Eq. (2.8), then clearly

$$\frac{\partial \nu}{\partial \rho} = \rho \sum_{i=1}^{3} L_i r_i^{-3} \tag{2.10}$$

$$\frac{\partial \nu}{\partial z} = \sum_{i=1}^{3} L_i (z + \alpha_i) r_i^{-3}$$

$$\begin{aligned}\frac{\partial \lambda}{\partial \rho} &= \rho \sum_{i,j=1}^{3} L_i L_j [\rho^2 - (z + \alpha_i)(z + \alpha_j)] r_i^{-3} r_j^{-3} \\ &= \rho \sum_{i=1}^{3} L_i^2 [\rho^2 - (z + \alpha_i)^2] r_i^{-6} \\ &\quad + \rho \sum_{\substack{i,j=1 \\ i \neq j}}^{3} L_i L_j [\rho^2 - (z + \alpha_i)(z + \alpha_j)] r_i^{-3} r_j^{-3}\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\frac{\partial \lambda}{\partial z} &= 2\rho^2 \sum_{i=1}^{3} L_i^2 (z + \alpha_i) r_i^{-6} \\ &\quad + 2\rho^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{3} L_i L_j [(z + \alpha_i) + (z + \alpha_j)] r_i^{-3} r_j^{-3}\end{aligned} \tag{2.12}$$

After some manipulation it is a relatively straightforward matter to show that integration of Eq. (2.11) with respect to ρ yields

$$\begin{aligned}\lambda = &-\tfrac{1}{2} \sum_{i=1}^{3} \rho^2 L_i^2 r_i^{-4} \\ &-\tfrac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} L_i L_j [(r_i r_j)^{-1} - (\alpha_i - \alpha_j)^{-2} (r_i r_j^{-1} + r_i^{-1} r_j)] + A(z)\end{aligned} \tag{2.13}$$

Differentiating Eq. (2.13) with respect to z and comparing the result with Eq. (2.12) yields

$$A(z) = \text{const} \tag{2.14}$$

The constant (2.14) is chosen so as to make the metric Minkowskian at infinity, thus

$$A = - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} L_i L_j (\alpha_i - \alpha_j)^{-2} \tag{2.15}$$

The complete expression for λ is then

$$\lambda = -\tfrac{1}{2}\sum_{i=1}^{3} \rho^2 L_i^2 r_i^{-4} - \tfrac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3} L_iL_j[(r_ir_j)^{-1} - (\alpha_i - \alpha_j)^{-2}(r_ir_j^{-1} + r_i^{-1}r_j) + 2(\alpha_i - \alpha_j)^{-2}] \tag{2.16}$$

2.3 *The Relationship between L_1, L_2, and L_3*

In order that this solution represent three isolated masses lying on the z axis, not only must ν and λ and their first derivatives be continuous, but λ must vanish everywhere for $\rho = 0$, except at the mass points. This latter condition is necessary in order that the infinitesimal circle in the plane z = const, t = const with center $\rho = 0$ will have the ratio of circumference to diameter equal to π^2. By considering Eq. (2.16) and the fact that

$$r_ir_j \sin\theta_{ij} = \rho\,|(\alpha_i - \alpha_j)| \tag{2.17}$$

where θ_{ij} is the angle subtended by r_i and r_j, it is easily seen that

$$\lambda = -\tfrac{1}{2}\sum_{i=1}^{3} \rho^2 L_i^2 r_i^{-4} + \sum_{\substack{i,j=1\\ i\neq j}}^{3} L_iL_j(\cos\theta_{ij} - 1)(\alpha_i - \alpha_j)^{-2} \tag{2.18}$$

In order that $\lambda(\rho = 0, z) = 0$ then

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} L_iL_j(\alpha_i - \alpha_j)^{-2}(\cos\theta_{ij} - 1)\,|_{\rho=0} = 0 \tag{2.19}$$

Clearly at all points on the z axis and outside the grouping of the three mass centers $\theta_{ij} = 0$, and Eq. (2.19) is satisfied.

However, to satisfy Eq. (2.19) for all points within the grouping of the three mass centers, we must impose the further condition

$$\frac{L_i}{L_j} = (-)^{i+j}\frac{(\alpha_i - \alpha_k)^2}{(\alpha_j - \alpha_k)^2}, \qquad i, j, k = 1, 2, 3;\ i \neq j \neq k \neq i \tag{2.20}$$

Equation (2.20) is just the condition for the equilibrium of two positive (or negative) masses held apart by the counterbalancing repulsion of an intermediary negative (or positive) mass.

Equations (2.8), (2.16), and (2.20) thus complete the solution.

3. *A Special Case*

If we now look to the situation of two equal masses m situated at $z \pm a$ and a mass of $-m/4$ situated at the origin, then in Newtonian theory and in general relativity they will all lie at rest in equilibrium.

In general relativity the metric is then given by Eq. (2.1), where

$$\nu = -L\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{4r}\right)$$

$$\lambda = -\frac{\rho^2 L^2}{2}\left(\frac{1}{r_1{}^4} + \frac{1}{r_2{}^4} + \frac{1}{16r^4}\right) - \frac{L^2}{4a^2}\left[\frac{4a^2}{r_1 r_2} - \frac{a^2}{r_1 r} - \frac{a^2}{r_2 r} - \left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - \frac{r}{r_1} - \frac{r_1}{r} - \frac{r}{r_2} - \frac{r_2}{r} + 2\right)\right] \tag{3.1}$$

where

$$r_1{}^2 = \rho^2 + (z+a)^2, \qquad r_2{}^2 = \rho^2 + (z-a)^2, \qquad r^2 = \rho^2 + z^2$$

and

$$L = Gmc^{-2}$$

If we consider the limit, as $a \to 0$, of both ν and λ, we find that after a repeated application of L'Hôpital's rule

$$\nu = -L'r^{-1}, \qquad \lambda = -\rho^2 L'^2/2r^4 \tag{3.2}$$

where

$$L' = 7L/4$$

This is the single-center solution for a mass equal to the algebraic sum of the masses m, m, and $-m/4$, in agreement with Eqs. (2.6) and (2.7).

4. *Discussion*

While it has long been known that Newtonian theory admits the possibility of negative mass and also the combination of negative and positive mass, it was not clear that the latter situation could find its parallel in Einstein's general theory of relativity. As Bondi notes (1957), the integration that gives the Schwarzschild solution produces a constant that is identified with the mass of an isolated spherically symmetric particle. The sign of the mass can be chosen as positive or negative so as to correlate with position or negative mass in Newtonian theory. This paper, however, presents for the first time a complete solution involving masses of both signs in general relativity in direct correspondence with the Newtonian picture.

Consistency with previous results is also demonstrated by considering the limit, as the separation parameter goes to zero, of a three-center situation that is symmetric about the origin.

It should also be noted that this work will quite easily extend to n bodies lying on the z axis such that n is an odd number and the masses are alternately positive and negative, being of suitable size so as to lie in equilibrium.

References

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